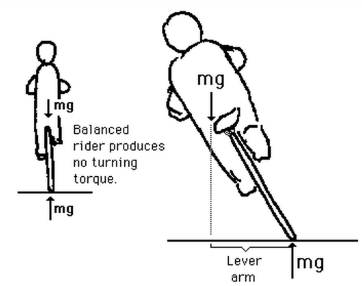
Lecture 9 - Rotational Dynamics

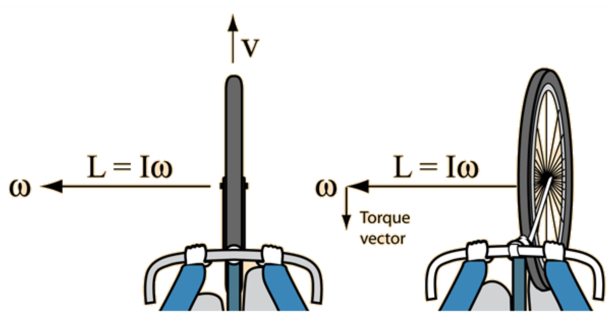
A Puzzle...

Angular momentum is a 3D vector, and changing its direction produces a torque $\vec{\tau} = \frac{d\vec{L}}{dt}$. An important application in our daily lives is that bicycles don't fall over when you turn. Explain what happens when you lean over.



Solution

In the diagram to the left, as the wheel turn there is angular momentum (about the contact point with the ground) pointing to the left. When you lean over on a bicycle the angular momentum points left and downwards, and there is a torque $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\,\vec{g}$ which points out of the page. Since $\vec{\tau} = \frac{d\vec{L}}{dt}$, the angular momentum vector will rotate around and cause the bike to turn.

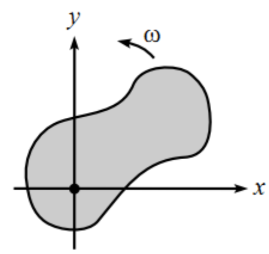


The full story is a lot more complicated, and the physics of bicycles is not completely understood. Indeed, there is more to the story then what was discussed above. For example, if the wheels are not spinning, then we all agree that the leaning on a bicycle will cause it to tip over; the above analysis indicates that we need to consider the angular momentum of the entire bike + rider system (and not just the wheels). Walter Lewin gives a neat demonstration of a related phenomena with just one wheel. □

Moment of Inertia

Basics

Suppose we have a (flat) pancake object in the x-y plane rotating about an axis $\vec{\omega} = \omega\,\hat{z}$.



Consider a little piece of the body, with mass dm and position vector $\vec{r} = \langle x, y \rangle$ whose magnitude equals $r = \sqrt{x^2 + y^2}$. This little piece travels in a circle around the origin with speed $v = \omega\, r$ and therefore has angular momentum (relative to the origin) $\vec{L} = \vec{r} \times \vec{p} = (r\, v\, dm)\, \hat{z} = \left(r^2\, \omega\, dm\right) \hat{z}$. Thus, the angular momentum of the entire body equals

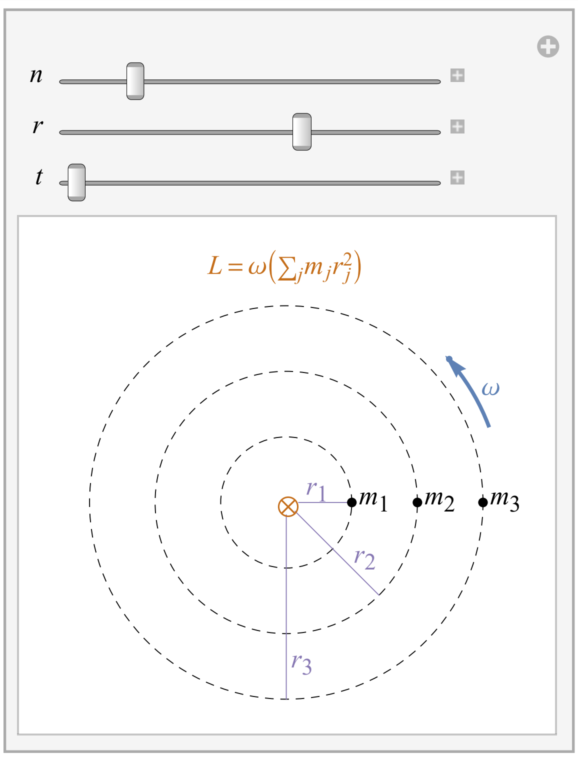
$$\begin{aligned} \vec{L} &= \int (r^2\, \omega\, dm)\, \hat{z} \\ &= \omega\, \hat{z} \int r^2\, dm \\ &\equiv I\, \omega\, \hat{z} \end{aligned} \tag{1}$$

where we have defined the *moment of inertia*

$$I = \int r^2\, dm \tag{2}$$

in the z-direction. For a collection of point masses in the x-y plane, the moment of inertia would take the form

$$I = \sum_j m_j\, r_j^2 \tag{3}$$



The *moment of inertia* is calculated *about an axis*, distinguishing it from *torque* and *angular momentum* which are calculated *about a point*. In this class, we will typically confine ourselves to objects that are either flat pancakes (like a circle in the x-y plane) or are stretched out uniformly in the z-direction (such as a cylinder whose axis lies in the z-direction). In the latter case, we can just pretend that the cylinder is really a flat pancake in the x-y plane, because as long as we keep its mass constant its moment of inertia will not change.

The angular momentum and torque (about the origin) equal

$$L = I\,\omega \tag{4}$$

$$\tau = \frac{dL}{dt} = I\,\alpha \tag{5}$$

where $\alpha \equiv \frac{d\omega}{dt}$ is the angular acceleration of the object.

Internal Forces

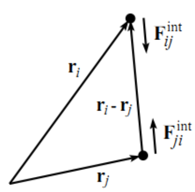
Throughout our discussion of torques, we have always ignored internal forces. The next problem shows why this isn't a problem.

Example

Given a collection of particles with positions $\vec{r}_j$, let the force on the j^{th} particle, due to all the others, be $\vec{F}_j^{\text{int}}$. Assuming that the force between any two particles is directed along the line between them, use Newton's third law to show that $\sum_j \vec{r}_j \times \vec{F}_j^{\text{int}} = 0$.

Solution

Let $\vec{F}_{ij}^{\text{int}}$ be the force that the i^{th} particle feels from the j^{th} particle so that $\vec{F}_j^{\text{int}} = \sum_k \vec{F}_{jk}^{\text{int}}$.



The total internal torque equals

$$\vec{\tau}_{\text{int}} = \sum_j \vec{r}_j \times \vec{F}_j^{\text{int}} = \sum_{j,k} \vec{r}_j \times \vec{F}_{jk}^{\text{int}} \tag{6}$$

Switching the labels between j and k, and using Newton's 3rd law $\vec{F}_{jk}^{\text{int}} = -\vec{F}_{kj}^{\text{int}}$, we find

$$\vec{\tau}_{\text{int}} = \sum_{j,k} \vec{r}_k \times \vec{F}_{kj}^{\text{int}} = -\sum_{j,k} \vec{r}_k \times \vec{F}_{jk}^{\text{int}} \tag{7}$$

Adding up these two equations,

$$2\,\vec{\tau}_{\text{int}} = \sum_{j,k} (\vec{r}_j - \vec{r}_k) \times \vec{F}_{jk}^{\text{int}} = 0 \tag{8}$$

where in the last step we have used the fact that $\vec{r}_j - \vec{r}_k$ is parallel to $\vec{F}_{jk}^{\text{int}}$. Note that the idea behind this argument is that the torques cancel in pairs. This is clear from the diagram above, since the forces are equal and opposite and have the same lever arm. □

Center of Mass

Consider a group of particles with masses m_j are positions $\vec{r}_j$. We define the *center of mass* $\vec{R}_{\text{CM}}$ as

$$\vec{R}_{\text{CM}} = \frac{\sum_j m_j \vec{r}_j}{M} \tag{9}$$

where

$$M = \sum_j m_j \tag{10}$$

represents the total mass. Differentiating the center of mass, we find the velocity and acceleration of the center of mass,

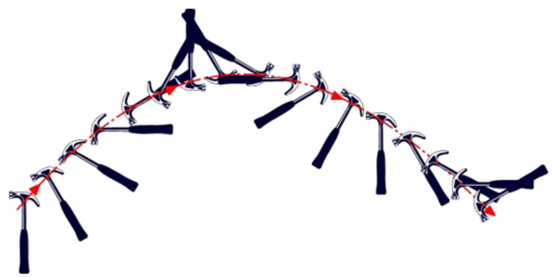
$$\vec{V}_{\text{CM}} = \frac{d\vec{r}_{\text{CM}}}{dt} = \frac{\sum_j m_j \vec{v}_j}{M} \tag{11}$$

$$\vec{A}_{\text{CM}} = \frac{d\vec{v}_{\text{CM}}}{dt} = \frac{\sum_j m_j \vec{a}_j}{M} \tag{12}$$

The total force on all of the particles is given by

$$\vec{F}_{\text{tot}} = M\vec{A}_{\text{CM}} \tag{13}$$

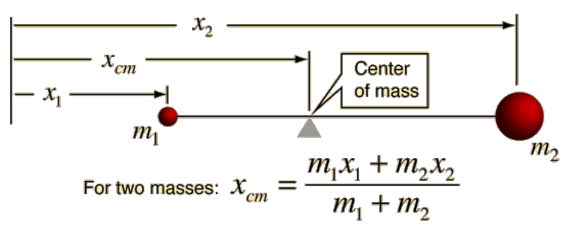
Recall from the previous section that a rigid body is a collection of point masses, and Equation (13) hold regardless of whether this rigid body is rotating or not. For example, consider the trajectory of the hammer shown below. Although it complete motion is complex, the net force acting on this object is due solely to gravity. Thus, the trajectory of the center of mass is simple 2D projectile motion, and hence the center of mass will arc in a parabola, as shown by the red curve.



In this class, we will treat all extended bodies as rigid objects (i.e. we assume they never deform), with the sole exception being inelastic collisions where two objects will stick together.

Example

The center of mass of m_1 and m_2 is shown below relative to the vertical axis on the left. Find the center of mass relative to the position of m_1. Do the two locations agree?



Solution

Relative to m_1, the position of m_1 is 0 and the position of m_2 is $x_2 - x_1$. Using Equation (9), the center of mass will be to the right of m_1 by a distance

$$R_{\text{CM}}^{(m_1)} = \frac{m_2(x_2 - x_1)}{m_1 + m_2} \tag{14}$$

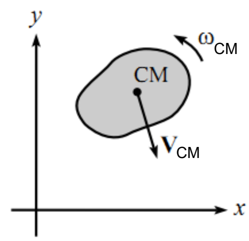
Relative to the axis on the left (a distance x_1 to the left of m_1), the distance to the center of mass would be

$$R_{\text{CM}}^{(\text{left axis})} = x_1 + \frac{m_2(x_2 - x_1)}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{15}$$

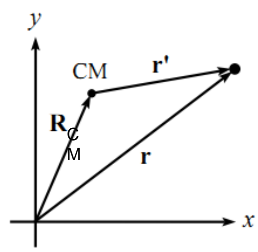
The two answers match, as they must, since the location of the center of mass is independent of your starting point for using Equation (9). □

Angular Momentum and Kinetic Energy Decouple about the Center of Mass

What is the most general motion of a (flat) pancake object constrained to lie in the x-y plane? The object can be both translating and rotating. For this problem, it is advantageous to work in the center of mass frame.



Let the position vector of the center of mass be $\vec{R}_{\mathrm{CM}} = \langle X_{\mathrm{CM}}, Y_{\mathrm{CM}} \rangle$ with velocity $\vec{V}_{\mathrm{CM}}$. In the center of mass frame, any point on the object will have the relative position $\vec{r}\,' = \langle x', y' \rangle$ so that in the inertial (lab) frame the position of this point will be $\vec{r} = \langle x, y \rangle$ satisfying $\vec{r} = \vec{R}_{\mathrm{CM}} + \vec{r}\,'$



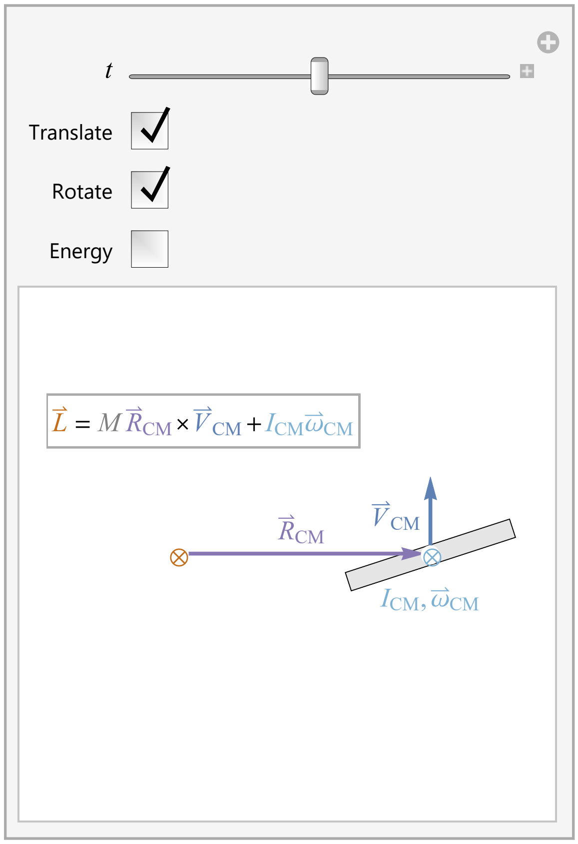
Let the relative velocity of the point $\vec{r}\,'$ in the center of mass frame equal $\vec{v}\,'$, so that in the lab frame its velocity equals $\vec{v} = \vec{V}_{\mathrm{CM}} + \vec{v}\,'$. Let the object rotate with angular velocity ω_{CM} about the center of mass (in both the lab frame and center of mass frame). Then $v' = \omega_{\mathrm{CM}}\, r'$ and the angular momentum about the origin equals

$$\begin{aligned}
\vec{L} &= \int \vec{r} \times \vec{v}\, dm \\
&= \int \left(\vec{R}_{\mathrm{CM}} + \vec{r}\,'\right) \times \left(\vec{V}_{\mathrm{CM}} + \vec{v}\,'\right) dm \\
&= M\, \vec{R}_{\mathrm{CM}} \times \vec{V}_{\mathrm{CM}} + \int \left(\vec{R}_{\mathrm{CM}} \times \vec{v}\,' + \vec{r}\,' \times \vec{V}_{\mathrm{CM}}\right) dm + \int \vec{r}\,' \times \vec{v}\,'\, dm \\
&= M\, \vec{R}_{\mathrm{CM}} \times \vec{V}_{\mathrm{CM}} + \int (r')^2\, \omega_{\mathrm{CM}}\, \hat{z}\, dm \\
&= M\, \vec{R}_{\mathrm{CM}} \times \vec{V}_{\mathrm{CM}} + I_{\mathrm{CM}}\, \omega_{\mathrm{CM}}\, \hat{z}
\end{aligned} \tag{16}$$

where in the third line we used the fact that in the center of mass frame, $\int \vec{v}\,'\, dm = \int \vec{r}\,'\, dm = 0$ so that the cross terms vanish. The quantity I_{CM} equals the moment of inertia of the object in the z-direction going through the center of mass of the object.

This result is incredible and is definitely worth knowing. In short, it states that:

The angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the center of mass and finding the angular momentum of this point mass relative to the origin, and adding on the angular momentum of the body relative to the center of mass. (17)



Example

For example, the moon orbiting Earth has two types of angular momentum: one from its circular orbit around the Earth and the other from its rotation about its own axis. Compute the magnitudes of both of these angular momenta. (Note that these angular momentum vectors may point in different directions, but in this problem we ignore that because we don't add them together.)

Solution

We define the following quantities:

- M = mass of the moon
- R_{CM} = vector from the center of the Earth to the moon
- V_{CM} = velocity of the center of mass of the moon
- ω_{CM} = angular velocity of the moon about its own axis
- I_{CM} = moment of inertia about the center of mass of the moon

Using Equation (16),

$$L_{\text{orbit}} = M R_{\mathrm{CM}} V_{\mathrm{CM}} \tag{18}$$

$$L_{\text{rotation}} = I_{\mathrm{CM}} \omega_{\mathrm{CM}} \tag{19}$$

Note that $V_{\mathrm{CM}} \neq R_{\mathrm{CM}} \omega_{\mathrm{CM}}$, since V_{CM} refers to the speed of the moon's center of mass while ω_{CM} is the angular velocity of the moon spinning about its axis. The correct statement is that given a vector $\vec{r}\,'$ originating from the moon's center of mass, its velocity about the moon's center of mass is given by $\vec{\omega}_{\mathrm{CM}} \times \vec{r}\,'$. □

Now let's calculate the kinetic energy of the object,

$$\begin{aligned}
\mathrm{KE} &= \int \frac{1}{2} v^2 \, dm \\
&= \int \frac{1}{2} \vec{v} \cdot \vec{v} \, dm \\
&= \int \frac{1}{2} \left(\vec{V}_{\mathrm{CM}} + \vec{v}'\right) \cdot \left(\vec{V}_{\mathrm{CM}} + \vec{v}'\right) dm \\
&= \frac{1}{2} M V_{\mathrm{CM}}^2 + \int \vec{V}_{\mathrm{CM}} \cdot \vec{v}' \, dm + \int \frac{1}{2} (v')^2 \, dm \\
&= \frac{1}{2} M V_{\mathrm{CM}}^2 + \int \frac{1}{2} (r')^2 \, \omega_{\mathrm{CM}}^2 \, dm \\
&= \frac{1}{2} M V_{\mathrm{CM}}^2 + \frac{1}{2} I_{\mathrm{CM}} \, \omega_{\mathrm{CM}}^2
\end{aligned}$$

where in the 5th line we used the fact that $\int \vec{v}' \, dm = 0$ in the center of mass frame. This is another beautiful result, which can be summarized as:

The kinetic energy of a body can be found by treating the body as a point mass located at the center of mass, and adding on the kinetic energy of the body due to the motion relative to the center of mass. (21)

Analogies between Translational and Rotational Motion

The equations in rotational dynamics are very similar to those in linear dynamics that we have already seen provided that you make the following identifications

$$\begin{array}{ccc}
\text{Translation} & \leftrightarrow & \text{Rotation} \\
x & \leftrightarrow & \theta \\
m & \leftrightarrow & I \\
\vec{F} & \leftrightarrow & \vec{\tau} \\
\vec{p} & \leftrightarrow & \vec{L}
\end{array} \tag{22}$$

For example, the acceleration $a = \frac{d^2x}{dt^2}$ translates into an angular acceleration $\alpha = \frac{d^2\theta}{dt^2}$. The kinetic energy of purely translational motion $\frac{1}{2} m \, v^2$ translates into the energy of purely rotational motion $\frac{1}{2} I \, \omega^2$.

Supplemental Section: Work and Power

Recall that the *work* done by a force $\vec{F}$ on an object that travels along a curve C equals

$$W = \int_C \vec{F} \cdot \vec{v} \, dt = \int_C \vec{F} \cdot d\vec{x} \tag{23}$$

where $\vec{x}$ defines the curve C and $\vec{v}$ is the velocity along this curve.

The *power* at any point along the curve C equals

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \tag{24}$$

For purely rotational systems, power equals

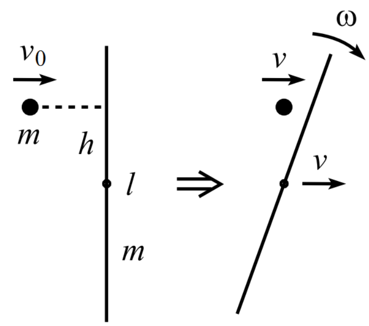
$$P = \vec{\tau} \cdot \vec{\omega} \tag{25}$$

Collision Problems

With our new understanding of translational and rotational kinetic energy, we can now consider very general collision problems.

Example (Elastic collision)

A mass m travels with speed v_0 perpendicularly to a stick with the same mass m and length l, which is initially at rest. At what height h above the center of the stick should the mass collide elastically with the stick, so that the mass and the center of the stick move with equal speed v after the collision?



Solution

Conservation of momentum yields

$$m\, v_0 = m\, v + m\, v \tag{26}$$

which lets us solve for v,

$$v = \frac{v_0}{2} \tag{27}$$

Conservation of energy yields

$$\frac{1}{2} m\, v_0^2 = \frac{1}{2} m\, v^2 + \frac{1}{2} m\, v^2 + \frac{1}{2} I\, \omega^2 \tag{28}$$

Upon substituting $v = \frac{v_0}{2}$ from above, we can solve for ω,

$$\frac{1}{2} m\, v_0^2 = \frac{1}{2} m \left(\frac{v_0}{2}\right)^2 + \frac{1}{2} m \left(\frac{v_0}{2}\right)^2 + \frac{1}{2} I\, \omega^2 \tag{29}$$

$$\omega = \left(\frac{m}{2\, I}\right)^{1/2} v_0 \tag{30}$$

We compute angular about a fixed point O that coincides with the initial center of the stick. Conservation of angular momentum yields

$$m\, h\, v_0 = m\, h\, v + I\, \omega \tag{31}$$

where the final velocity of the stick's center of mass does not contribute because it passes through point O. Using $v = \frac{v_0}{2}$ from above, we find

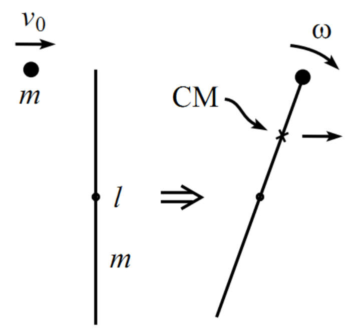
$$h = \left(\frac{2\, I}{m}\right)^{1/2} \tag{32}$$

As we will see next time, for the particular case of a uniform stick rotating about its center, $I = \frac{1}{12} m\, l^2$ and therefore $h = \frac{l}{\sqrt{6}}$. □

We can repeat this problem for the inelastic case, where we cannot conserve momentum. In such problems, we will have one less degree of freedom (because the objects are constrained to stick together) and therefore conservation of momentum and angular momentum will be enough.

Example (Inelastic collision)

A mass m travels with speed v_0 perpendicularly to a stick of mass m and length l, which is initially at rest. The mass collides completely inelastically with the stick at one of its ends, and sticks to it. What is the resulting angular velocity of the system?



Solution

The first thing to note is that the center of mass lies a distance $\frac{l}{4}$ from the end of the stick. Second, we will break the motion of the final object (the stick plus the mass at its end) into translational motion and rotation about the center of mass. We define $\tilde{I}_{CM}$ to be the moment of inertia of the final object.

Conservation of momentum tells us that the velocity v of the final object equals $v = \frac{v_0}{2}$. To use conservation of angular momentum, we first need to choose the point O about which we calculate angular momentum. While all choices will lead the same results, choosing wisely can make the calculations significantly simpler.

Solution 1

Choose point O as the fixed point which coincides with the center of mass right when the collision happens ($\frac{l}{4}$ from the end of the stick). Conservation of L yields

$$m\, v_0\, \frac{l}{4} = \tilde{I}_{CM}\, \omega \tag{33}$$

$$\omega = \frac{m\, v_0\, l}{4\, \tilde{I}_{CM}} \tag{34}$$

where the linear motion of the final object does not enter because it passes through point O.

Solution 2

Choose point O as the fixed point which coincides with the center of the stick. Conservation of L yields

$$m\, v_0\, \frac{l}{2} = (2\, m)\, v\, \frac{l}{4} + \tilde{I}_{CM}\, \omega \tag{35}$$

which upon substituting $v = \frac{v_0}{2}$ we again obtain

$$\omega = \frac{m\, v_0\, l}{4\, \tilde{I}_{CM}} \tag{36}$$

As we will see next time, the moment of inertia for the final object can be computed using the parallel axis theorem as $\tilde{I}_{CM} = I_{CM}^{stick} + I_{CM}^{mass} = \left(\frac{1}{12}\, m\, l^2 + m\left(\frac{l}{4}\right)^2\right) + m\left(\frac{l}{4}\right)^2 = \frac{5}{24}\, m\, l^2$. □

Rolling Problems

Mathematica Initialization